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Low-frequency azimuthal stability analysis of Hall thrusters

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This paper presents a linearized two-dimensional (axial and azimuthal) fluid model of the Hall thruster discharge with the goal of understanding the mechanism responsible for the azimuthal oscillations in the ionization region. After a short review of the linear stability analyses carried out within the Hall Thruster research community, our model is presented. It is is based on the one dimensional model of Ahedo *et al.*,[1] which has been extensively used in the past to characterize the plasma inside the thruster and the physics behind the ionization and acceleration regions of the thruster. Out of the myriad of oscillations in a Hall thruster, that of interest in this paper is the so-called spoke oscillation. Recent experiments[2–7] have shown the presence of the spoke in a rather large variety of Hall thrusters ranging from cylindrical to more conventional annular ones. However, there is not yet a clear understanding of the mechanism promoting and sustaining the spoke. Ultimately, the model presented here shall allow identifying the physics behind the spoke. The model can as well capture axial oscillations such as the breathing mode. Preliminary results from the model are shown for the more widely known and understood breathing mode.

Nomenclature

$_{i,e,n}$	subindex for ion, electron and neutral species
$_{x,y}$	subindex for axial and azimuthal coordinates
e	electron charge
m_e, m_i	electron mass and ion mass
$\vec{E}, \vec{B}$	electric and magnetic field
f_0	zero-th order solution of macroscopic variable f
$\hat{f}$	perturbation of macroscopic variable f
$\bar{f}$	coefficient of Fourier-like perturbation of variable f
$\tilde{f}$	non-dimensional version of variable f or coefficient $\bar{f}$
n, n_n	plasma and neutral density
$\vec{v}_j$	velocity vector of species j
v_{jx}, v_{jy}	axial and azimuthal velocity of species j
T_e	electron temperature
ϕ	electric potential
ν_i	ionization frequency
ν_e	electron-neutral collision frequency
ν_w, ν_{we}	particle and energy wall-loss frequencies
α_i	energy loss per actual ionization
$\omega_{i,e}$	ion and electron cyclotron Larmor frequencies
ω	frequency of perturbation
k	azimuthal wave number of perturbation
χ	Hall parameter
a_w	wall accommodation factor

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I. Introduction

The Hall Effect Thruster (HET) is a type of electric propulsion device initially developed in the 1960's by both the USA[8] and the former USSR[9,10] independently. The development continued in the shadow during the 1970's and 1980's in the USSR to reach a mature status. In the 1990's the advanced state of this Russian technology arrived to western countries, which rapidly restarted the analysis and development of Hall thrusters. Nowadays, there are several companies manufacturing modern Hall thrusters for operational use in USA, Russia and Europe. The main applications of these thrusters are: low-thrust propulsion of interplanetary probes, orbital raising of satellites and north-south station-keeping (NSSK) of geostationary satellites.[11]

Over the last decade great efforts have been dedicated to the understanding of the physics of Hall Effect Thrusters. However, there are still some important aspects to clarify. One of them is the so-called *anomalous diffusion*, responsible for an excessive electron axial current that cannot be explained with classical collisional theories. An externally imposed radial magnetic field and an axial electric field trap the electrons in an azimuthal closed-drift and, according to classical theory, the only mechanism that allows the electrons to drift axially along the channel is the collisions with other species. However, the effective electron conductivity measured experimentally is two orders of magnitude higher than that expected from collisions. Thus, another mechanism is suspected to enhance the electron mobility. One of the explanations for this is the presence of azimuthal oscillations with correlated density and electric field fluctuations both in the ionization and acceleration regions of the thruster. Several experiments have confirmed the presence of these oscillations using various experimental techniques. These oscillations are normally divided into low frequency (a few kHz), such as the spoke oscillation,[12] and high frequency (a few MHz) oscillations, such as the electron-drift wave,[13] as proposed by Choueiri.[14]

Several azimuthal local linear stability analyses have been carried out by the HET community.[13,15–29] However, many of these analyses focus on the acceleration region, where the ionization can be safely neglected, and on the high frequency regime. However, experimental results show the presence of azimuthal oscillations in the low frequency range located mostly in the ionization region of the thruster and appearing normally in the lower part of the current-voltage curve. In particular, the so-called spoke oscillation, using the same terminology proposed by Choueiri,[14] is one of these low frequency azimuthal oscillations. This oscillation was first detected by Janes and Lowder[12] in the 1960's and, more recently, was measured by Chesta *et al.*[30] with a modern laboratory Hall thruster both at low and high voltage. During the last two years several experiments[2–7] have shown the existence of this type of oscillation in various modern Hall thrusters with different size and operating conditions.

In a previous paper[31] a local linear stability analysis of the ionization region of the Hall thruster against small perturbations was presented. In that study a one dimensional (1D) model is used including three particle species: neutrals, singly-charged ions, and electrons. The formulation yields a simple algebraic eigenvalue problem that can be easily solved. The results indicate the existence of an instability that gives rise to an azimuthal oscillation with a mode number m=1 caused by the ionization. However, that study treats the problem from a local point of view, what is only valid when the length scales of the gradients of the plasma macroscopic variables in the axial direction are much larger than the typical dimensions of interest (i.e., homogeneous approximation). In this work, we present a continuation of that study, but in this case we overcome the issue of the locality. In order to account properly for the axial gradients, it is necessary to solve the full differential eigenvalue problem. To this end, an approach similar to the one proposed by Ahedo *et al.*[32,33] is used. This approach has been employed successfully in the past to analyze the breathing mode.[32–34] Here, contrary to the one-dimensional formulation of Ahedo *et al.*, the azimuthal direction must be included in the model in order to capture possible azimuthal oscillations. This implies that the model is two dimensional. Moreover, the axial variation of the steady-state solution as well as the linear perturbations are derived from the model. The model is designed to allow simulating the oscillations and check if there is a correlation between the density and electric field perturbations enhancing the electron conductivity in the Hall thruster.

The paper is organized as follows. In Section II a short review of the available information on low frequency oscillations is presented from the experimental and theoretical points of view, paying special attention to stability analyses. A review of the one-dimension model of Ahedo is presented in Section III. Section IV extends the formulation to two dimensions and derives the linearized equations to solve for the perturbations. Additionally, some results for axial oscillations are presented. Finally, Section V is dedicated to conclusions.

II. Review of frequency azimuthal oscillations in Hall thrusters

A. Experimental results for low frequency oscillations

The first experimental evidence of azimuthal oscillations in a Hall device was obtained by Janes and Lowder.[12] This experiment detected in a Hall accelerator a spoke by means of azimuthally separated Langmuir probes. The spoke consists of a density wave travelling in the azimuthal $E \times B$ direction with a phase velocity of a few km/s and an axial tilt of 20 degrees. The density oscillation is correlated with the oscillating electric field. As a consequence, a net axial electron current is generated from the oscillations enhancing, thus, the electron mobility within the thruster. Similar results were obtained shortly after by Lomas *et al.*[15]

More recently, Meezan and Capelli[35–37] use several low-frequency diagnosis methods to measure experimentally the electron mobility along the thruster. The results are in line with the general properties of anomalous diffusion presented above. In a separate study by Chesta *et al.*[30] two additional azimuthal low frequency waves with a tilt angle of 15-20 degrees and a mode number of $m = 1$ are detected one at low and another one at high voltage with properties similar to the spoke of Janes and Lowder.

In the last few years a significant number of experiments[2–7] have been carried out to detect the presence of the spoke and evaluate the influence of this oscillation on the electron mobility. Those experiments make use of high-speed imagery to detect the spoke travelling in the azimuthal direction and a segmented anode in the back of the channel to measure the electron current as a function of the azimuthal coordinate and time. Results[2,3] indicate that up a 50% of the electron current is transported through the high density region of the spoke in a cylindrical Hall thruster. In that case, the spoke travels in the $E \times B$ direction with a phase speed of about 2 km/s and wave number consistent with a $m = 1$ mode. Additionally, the influence of the cathode operation on the spoke and the electron mobility is analysed in that research. In particular, if the cathode emission is increased, the spoke no longer occurs and the electron conductivity is significantly reduced. McDonald and Gallimore[5,6] analyze several annular Hall thrusters detecting similar oscillations to those of Raitses *et al.*. Very recently, a method to control the spoke has been proposed[7] and successfully applied to a cylindrical Hall thruster. This method is based on similar principles to those used to control the breathing mode oscillation. It consists of a feedback control of the voltage of the anode segments. The results show the complete removal of the azimuthal oscillation and a decrease of 10% of the overall current.

B. Linear stability analyses for low and high-frequency azimuthal oscillations

The first complete linear stability analysis of the HET discharge was carried out by Esipchuk and Tilinin,[13] as a continuation of Morozov's analysis.[38] In Esipchuk's study, the linear stability of a collisionless two-fluid system (electrons and ions) is analysed in the electrostatic regime. Low frequency (LF) waves are predicted in locations where the gradient of the magnetic field to density ratio, B/n, is negative. However, the formulation does not include ionization which is claimed to be very important by Chesta.[16]

Also in the 1970's, Lomas *et al.*[15] carry out a simplified linear stability analysis of the Hall accelerator and suggest the growth of electro-thermal instabilities linked to the spokes detected experimentally. However, the growth mechanism is related to the interaction of ionization and volume recombination, the latter being negligible in modern HETs.

Chesta *et al.*[16] evaluate the linear stability of experimental steady-state profiles with a 2D three-fluid description of the discharge including ionization, particle collisions and electromagnetic effects. The main conclusion of that research is that the magnetic field and density gradients together with the ionization define the stability of LF azimuthal waves, but no analysis is carried out to unveil the exact mechanism.

Gallardo *et al.*[17] use a three-fluid formalism without electromagnetic terms to analyse the relation of the LF azimuthal waves with the electron anomalous diffusion. The main novelty with respect to previous studies is that the azimuthal three-fluid unsteady equations are solved to observe non-linear saturation effects. The conditions analysed in that case correspond to the ionization region of the channel and the results predict an $m = 3$ azimuthal wave travelling in the $-E \times B$ direction, which is opposite to what is normally measured.

Kapulkin[18] carries out yet another stability analysis specifically suited for the near-anode region where ionization is negligible and the temperature and the magnetic curvature contribute to the azimuthal drift of the electron flow. The results indicate the presence of an unstable wave of low frequency with a non-zero azimuthal component of the wave vector that can promote electron conductivity towards the anode.

Esipchuk and Tilinin[13] predict as well a high frequency (HF) wave of the electron-drift type. The inclusion of the electron inertia in the governing equations is the key to turn the dispersion relation into a cubic equation for the frequency. Roots of the cubic equation can appear in the high frequency range, close to the lower hybrid frequency, for certain combinations of the parameters of the model. A wave of similar properties is normally detected experimentally. However, Barral *et al.*[39] claim that the approach followed by Esipchuk *et al.* to account for the magnetic field gradient is not fully consistent and seems to invalidate the methodology of this and similar analyses.

In a series of studies,[19–21] Baranov *et al.* analyse separately the stability of the different regions of the thruster accounting for different effects: temperature terms and zero electric field in the ionization region and strong electric and magnetic fields in the acceleration region. The results predict azimuthal waves of high frequency both in the ionization and acceleration regions. In the case of the ionization region, the temperature seems to be the driving force. However, there is no experimental evidence of such high frequency azimuthal waves in the rear part of the thruster. Moreover, no ionization terms and neutral gas equations are considered in the model and therefore, low frequency phenomena are not taken into account.

Litvak *et al.*[22] focus on the effect of the electron collisions on the stability of the discharge. In that research, two types of unstable azimuthal oscillations appear: an electrostatic instability and an electromagnetic one. Both are promoted by the electron collisionality, but the former rotates with a frequency close to the lower hybrid frequency (10 MHz) while the latter propagates at the Alfven speed with a frequency of 1 MHz. According to that analysis, the higher growth rate of the electrostatic oscillation makes it more likely.

A formalism to analyse the linear stability of a two-fluid electrostatic model of the Hall discharge is presented by Thomas *et al.*[40,41] The result in the no-gradients limit coincides with the expression of Baranov *et al.*,[19] but the analysis is extended to account for electron-drift shear, electron-neutral collisions and electron temperature by means of an equation of state. The resulting dispersion relation is applied to the experimental profiles of a laboratory thruster and the results indicate the presence of unstable oscillations in the high frequency regime above 1 MHz.

Spektor[23] uses a linear stability method including collisions, macroscopic gradients and temperature terms to derive a dispersion relation that is used in turn to compute the equivalent anomalous electron collision frequency from plasma susceptibility. The results predict an anomalous diffusion scaling as $1/B^4$. Clearly, this relation is against the experimentally verified scaling law for the Bohm diffusion of $1/B$.[42]

On the other hand, Ducrocq *et al.*[24,25,43] use a kinetic model to describe the electron species. In this research an electron drift instability of short wave-length, close to the electron Larmor radius (1 mm), and high frequency (1-10 MHz), is predicted as a result of a resonance between the gyro-motion of the electrons and oscillations of the electric field.

A second approach to study the stability of the discharge involves analysing the eigenvalue problem obtained when the axial coordinate is not resolved in the usual Fourier form (i.e., $\exp(ik_x x)$). This method is followed by Kapulkin *et al.* in several studies[26–28] as well as by Litvak *et al.* in.[29] The same approach is used in classical fluid mechanics to study the well known Rayleigh instability. The main advantage of this method over the previous one is that the former method is based on a local analysis, while with the latter approach the stability of the discharge is analysed globally. This is precisely the method used in this work.

Kapulkin *et al.* in[26,27] and Litvak *et al.* in[29] conclude separately that Rayleigh-type instabilities appear inside the Hall thruster with frequencies *higher* than the lower hybrid frequency. In both researches, the phase speed of the unstable wave is of the order of the ratio E_{max}/B_{max}. Moreover, the exact value of the wave velocity coincides with the electron drift at a given point along the thruster. The influence of the density and magnetic field gradients is highlighted by both authors: the higher the gradients are, the lower the frequency of the wave is. As a consequence of the instability, the electric potential fall is spread out along the thruster reducing the maximum electric field. However, it is important to remark that these analyses focus on a frequency range above the one where high frequency oscillations are detected experimentally since the hybrid lower frequency is in the 1-10 MHz range for HETs.[14]

In a later study,[28] Kapulkin *et al.* analyse waves in the lower-hybrid frequency range with the same approach. The main novelty with respect to the previous formulation is the inclusion of the ion dynamics. As a result, a resonant effect with the lower hybrid frequency appears in the dispersion equation. The main driver for the instability is the same as before, the electron drift velocity non-uniformity. This theoretical instability has several properties in common with those measured experimentally: azimuthal propagation in the direction of the electron drift, frequency of 1 MHz and wavelength of the order of the channel circumference.

III. Review of the axial 1D model

The hypothesis and equations of the 1D model of Ahedo et al.[1] of the Hall discharge are reviewed in this section. Each one of the species present in a HET (neutrals, electrons and ions) is accounted for with a separate set of macroscopic fluid equations based on different conservation principles: particle, momentum and energy conservation laws. Quasi-neutrality is assumed as the Debye length in Hall thrusters is much smaller than the typical length scales of interest. This implies that Poisson's equation for the electric field is replaced with the condition of equal ion and electron particle densities. Furthermore, while electrons are highly magnetized, ions are considered to be unmagnetized since the ion Larmor radius is much larger than the typical thruster dimensions. At the same time, ions and neutrals are modelled as cold species, this is, no temperature is included for them in the equations and no energy equation is needed for those two species. On the other hand, due to the very low mass of the electrons, inertial effects are neglected in the electron momentum equation, yielding a diffusive model. Both electron-neutral collisions and ionization are introduced in the formulation. Wall energy losses and wall particle recombination are included in the model by means of equivalent frequencies. The electron kinetic energy is removed from the energy equation as it is considered negligible compared to the internal energy in most regions of the Hall Thruster. A sink of energy is introduced in the energy equation to account for the ionization and radiation losses. Heat conduction is neglected in the model as it is believed it does not play a major role in the azimuthal oscillations. This is in line with the model of Barral *et al.*[34,44–46] for the breathing mode, where no heat conduction is considered. Notice that the induced magnetic field is neglected and thus, Maxwell's equations are not included in the model. The magnetic field is equal to the field externally applied, which is stationary ($\partial\vec{B}/\partial t = 0$), solenoidal ($\nabla\cdot\vec{B} = 0$), irrotational ($\nabla\times\vec{B} = 0$) and radial. Consequently, the electric field derives from a potential ($\nabla\phi = -\vec{E}$). The resulting formulation may be written as:

$$
\begin{aligned}
\frac{d}{dx}(nv_{ex}) = \frac{d}{dx}(nv_{ix}) &= -\frac{d}{dx}(n_n v_{nx}) = n(\nu_i - \nu_w) \\
m_i n_n v_{nx}\frac{dv_{nx}}{dx} &= n_e\nu_w(1-a_w)(v_{ix}-v_{nx}) \\
m_i n v_{ix}\frac{dv_{ix}}{dx} &= -en\frac{d\phi}{dx} - m_i n\nu_i(v_{ix}-v_{nx}) \\
0 &= -\frac{d}{dx}(nT_e) + en\frac{d\phi}{dx} - m_e n\nu_e\chi^2 v_{ex} \\
\frac{\partial}{\partial t}\left(\frac{3}{2}nT_e\right) + \frac{d}{dx}\left(\frac{5}{2}nT_e v_{ex}\right) &= env_{ex}\frac{d\phi}{dx} - n\nu_i\alpha_i - n\nu_{we}T_e
\end{aligned}
\tag{1}
$$

where n_n and n are the neutral and plasma particle densities; v_{nx}, v_{ex} and v_{ix} are the fluid velocities of neutrals, electrons and ions respectively; T_e and ϕ are the electron temperature and electric potential; ν_e, ν_i, ν_w, ν_{we} represent the frequencies for electron-neutral collisions (including as well Bohm-type diffusion), ionization, particle wall recombination and energy wall losses respectively; a_w is the accommodation factor of the ions impacting the walls. The azimuthal momentum equation, $v_{ey} = v_{ex}\chi$ where $\chi = eB/m_e\nu_e \gg 1$ is the Hall parameter, has been used implicitly in the axial electron momentum equation.

An important aspect to take into account is the presence of sonic points. Since heat conduction is not considered, the sonic point condition is $T_e/m_i = (3/5)v_{ix}^2$. The combination of the previous equations allows obtaining an equation for $h_e \equiv \ln n_e$ as:

$$
Pv_{ex}\frac{dh_e}{dx} = G \tag{2}
$$

where $P = T_e/m_i - (3/5)v_{ix}^2$ and G is a function of the macroscopic variables but not of their derivatives.

It is possible to proof that there are two sonic point points in the domain, one singular ($G \neq 0$) in the anode sheath transition and another one regular ($G = 0$) inside the channel. The presence of these sonic points makes the resolution of the previous equations complex. A detailed description of the boundary conditions associated to the previous system of ordinary differential equations is described by Ahedo *et al.*[1] Expressions for the different frequencies mentioned above (ν_i,ν_e,ν_w,ν_{we}) may be found elsewhere.[1,47–49]

Various versions of the model above including different terms have been used to characterize the Hall discharge[1] , include heat conduction effects[47] , evaluate the influence of the wall losses[48] , carry out parametric investigations on the operating conditions[49] , model two-stage Hall thrusters[50] and even analyze the stability of the discharge against small axial perturbations to study the properties of the breathing mode[32–34] .

IV. Linearized two-dimensional model

A. General 2D formulation

The equations presented in the previous section can be extended to two dimensions (azimuthal and axial) with unsteady terms. Under the same hypothesis, the governing 2D unsteady equations of the plasma discharge may be written as:

$$
\begin{aligned}
\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}_e) = \frac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}_i) = -\frac{\partial n_n}{\partial t} - \nabla\cdot(n_n\vec{v}_n) = n(\nu_i - \nu_w) \\
m_i n_n\left(\frac{\partial \vec{v}_n}{\partial t} + \vec{v}_n\cdot\nabla\vec{v}_n\right) = n_e\nu_w(1-a_w)(\vec{v}_i - \vec{v}_n) \\
m_i n\left(\frac{\partial \vec{v}_i}{\partial t} + \vec{v}_i\cdot\nabla\vec{v}_i\right) = -en\nabla\phi - m_i n\nu_i(\vec{v}_i - \vec{v}_n) \\
0 = -\nabla(nT_e) - en(-\nabla\phi + \vec{v}_e\times\vec{B}) - m_e n\nu_e\vec{v}_e \\
\frac{\partial}{\partial t}\left(\frac{3}{2}nT_e\right) + \nabla\cdot\left(\frac{5}{2}nT_e\vec{v}_e\right) = en\vec{v}_e\cdot\nabla\phi - n\nu_i\alpha_i - n\nu_{we}T_e
\end{aligned}
\tag{3}
$$

It is important to remark that even though the equations above can be applied to the fully three-dimensional problem, in this case the radial variation of the variables is neglected reducing the problem to two-dimensions: axial (x) and azimuthal (y). Moreover, curvature effects in the azimuthal direction are also neglected as the mean radius of the channel is typically larger than the difference between the outer and inner radii. Obviously in the limit of a stationary and axi-symmetric solution the system of equations in (3) reduces to (1).

The equations in (3) can be rewritten in a form more amenable to the current study. To this end, the partial derivatives with respect to the time and the azimuthal coordinate are moved to the right hand side of the equations. In this manner, the left hand side resembles the equations in (1). The resulting equations can be combined in order to obtain an equation for the variable $h_e \equiv \ln n_e$ as:

$$
P v_{ex}\frac{\partial h_e}{\partial x} = G + G_t + G_y \tag{4}
$$

where $P = T_e/m_i - (3/5)v_{ix}^2$, G is a function of the macroscopic variables, but not of their derivatives, and identical to one derived for the one-dimensional model, G_t is a function of the macroscopic variables and proportional to their time derivatives, and G_y is a function of the macroscopic variables and proportional to their azimuthal derivatives.

There are two equations from (3) that deserve more attention. These are the azimuthal momentum equations for electrons and ions that may be expressed as:

$$
0 = -\frac{T_e}{m_i}\frac{\partial h_e}{\partial y} - \frac{\partial}{\partial y}\frac{T_e}{m_i} + \frac{e}{m_i}\frac{\partial\phi}{\partial y} + \frac{m_e}{m_i}\nu_e(\chi v_{ex} - v_{ey}) \tag{5}
$$

$$
v_{ix}\frac{\partial v_{iy}}{\partial x} = -\nu_i(v_{iy} - v_{ny}) - \frac{\partial v_{iy}}{\partial t} - \frac{e}{m_i}\frac{\partial\phi}{\partial y} - v_{iy}\frac{\partial v_{iy}}{\partial y} \tag{6}
$$

Equation (5) reduces to the condition $v_{ey} = \chi v_{ex}$ in the axi-symmetric limit. However, if there are azimuthal gradients, this equation explains why correlated density and electric field azimuthal oscillations may produce a net axial electron current.

Equation (6) has the peculiarity of defining another special point along the channel. In the point that separates the ionization region and ion back-streaming region ($v_{ix} = 0$), this equation requires a regular transition. Thus the right hand side of the equation must be zero at that point as well. This fact has important consequences on the way how the equations can be solved.

B. Linearization

The equations in (3) can be linearized around a steady-state and axi-symmetric solution (i.e., zero-th order solution). To this end it is possible to assume that a macroscopic variable $u(x, y, t)$ may be written as the sum of the zero-th order solution, $u_0(x)$, and a perturbation, $\tilde{u}_1(x, y, t)$. In order to consider consistently the axial gradients, the zero-th order solution and the coefficients of the Fourier expansion of the perturbation must retain the dependence on the axial coordinate. This is the main difference with respect to local stability analyses such as that presented previously by the authors[31] . The small perturbation hypothesis ($u_1 \ll u_0$) allows linearizing the equations in (3) and decouple the evolution equations for the zero-th order solution, which are given by the equations in (1), and for the perturbations, which consist of a linear system of differential equations. The perturbations may be expressed as:

$$u(x, y, t) = u_0(x) + \Re\left\{u_1(x; \omega; k) \exp(-i\omega t + iky)\right\} \tag{7}$$

where $\omega = \omega_r + i\omega_i$ is the frequency and k is the azimuthal wave number of the perturbation.

Similarly, the control parameters of the Hall discharge (voltage, mass flow, neutral velocity in the anode and the electron temperature at the cathode) may also be expanded in Fourier form:

$$w(t) = w_0 + \Re\left\{w_1 \exp(-i\omega t)\right\} \tag{8}$$

where $w_0 \gg w_1$ is the condition for small perturbations.

Applying the expression (7) to (3), it is possible to obtain the linear system of equations describing the evolution of the different perturbations along the channel. The resulting equations contain source terms proportional to the frequency, ω, and to the azimuthal wave number number, k, of the perturbation. These equations must be solved several times, once for each perturbation mode associated to the boundary conditions. Moreover, the zero-th order solution must also be solved together with the perturbed problem in order to be able to compute the coefficients of the perturbed equations. The equations can be integrated with any method suitable for ordinary differential equations.

The boundary conditions associated to the evolution equations are also derived from the linearization of the boundary conditions of the one dimensional problem. The result of this linearization has already been proposed by Ahedo *et al.*[32]

C. Solution method and self excited modes

The presence of a sonic point and zero ion-velocity point makes the integration process more complex. The solution is computed by concatenating the solutions obtained integrating the equations from the anode and the previous two points. Obviously, the solution must be continuous in these points and this imposes additional constraints to the final solution. Consequently, additional initial conditions are necessary so that the integration can be started from the anode and those two special points. In the end, the value of the weights of the perturbation modes in the final solution are obtained from the following algebraic system of equations:

$$AX = B \tag{9}$$

where X is a vector containing the weights of the perturbation modes, B is a vector with the coefficients of the linearized control parameters (zero for the pseudo control parameters as continuity is required) and A is a matrix with complex coefficients containing the partial derivatives of the control parameters with respect to the initial conditions of each of the perturbation modes. This matrix A depends on the zero-th order solution, w_0, as well as the frequency, ω, and the wave number, k, that is, $A = A(w_0, \omega, k)$.

In order to have self-excited modes, the previous algebraic system of equations must have non-trivial solutions for the homogeneous problem (i.e., case with $B = 0$). This is equivalent to:

$$\det A(w_0, \omega, k) = 0. \tag{10}$$

For each zero-th order solution given by the control parameters w_0 and for each azimuthal wave number k, equation (10) provides a condition to compute the frequency ω of the perturbation. If $\omega_i > 0$, then the perturbation is self-excited. In particular, $k = 0$ is the case with axial oscillations studied in the past for the analysis of the breathing mode[32–34] . In that case, the azimuthal momentum equations do not need to be solved, and thus, the additional complexity of the regular transition at the point with $v_{ix} = 0$ is avoided.

D. Preliminary results for axial oscillations

This section is devoted to the results obtained with the previous formulation in the case of axial oscillations. Similar results have been obtained in the past.[32–34] They are included here as way to validate the model presented above.

Figure 1 and 2 depict the profiles of the main macroscopic variables of the steady state solution (i.e., zero-th order solution) and the contour plots of the temporal evolution of the perturbations for a self excited mode. The control parameters have been selected for a typical SPT thruster. All variables in the plots are non-dimensional. In this simulation no wall effects are taken into account for simplicity. This fact, in combination with the fact that heat conduction is not considered in the model, make the temperature profile rather sharp, and as a consequence the ionization region is rather compact. Nevertheless, a growing oscillation is reproduced by the model. The oscillation corresponds to the breathing mode as the ionization front moves back and forth following the predator-prey scheme proposed originally by Fife[51] . While the neutral flow forms a forward travelling wave, the electron flow forms a backward travelling wave. More details about the breathing mode may be found in a series of analyses by Barral *et al.*[34, 44–46, 52] . In this paper, we are interested mainly in the ability of the model to reproduce the axial oscillation, as a first step to the modelling of azimuthal oscillations.

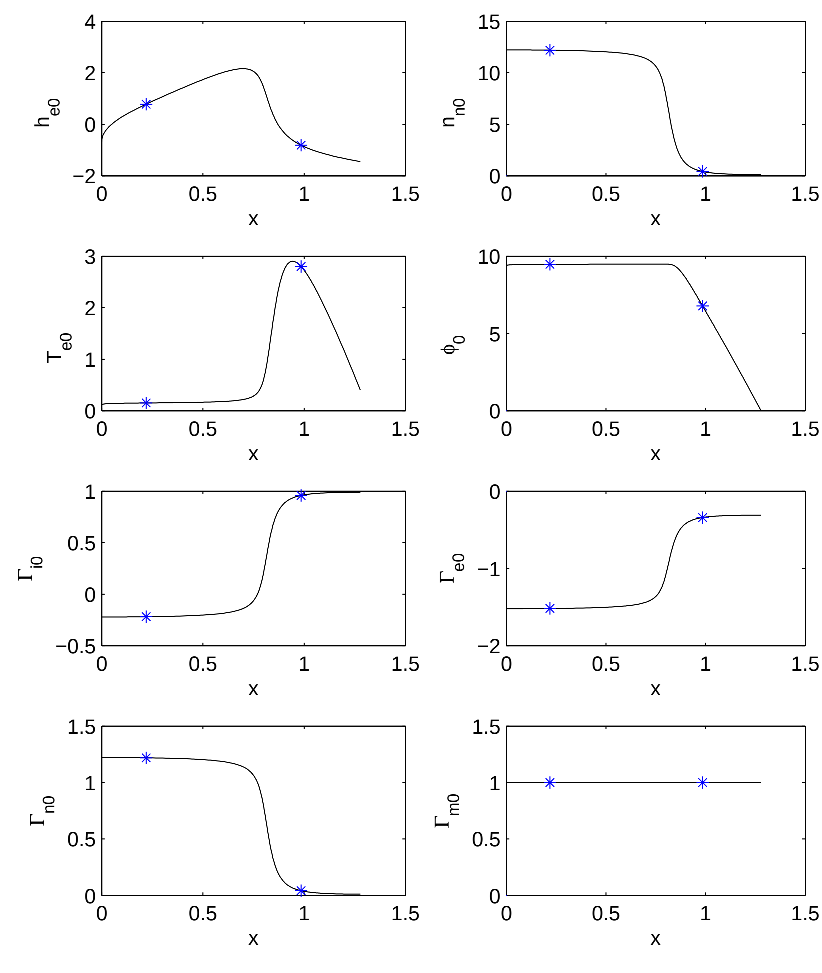

Figure 1. Zero-th order solution of a self-excited mode in non-dimensional variables. Axial coordinate x, plasma density n_0, neutral density n_{n0}, electron temperature T_{e0}, electric potential ϕ_0, ion flux Γ_{i0}, electron flux Γ_{e0}, neutral flux Γ_{n0}, mass flux Γ_{m0}. The asterisks represent the regular sonic point in the channel (subsonic-to-supersonic ion flow transition) and the point where the solutions from the anode and the sonic point are matched)

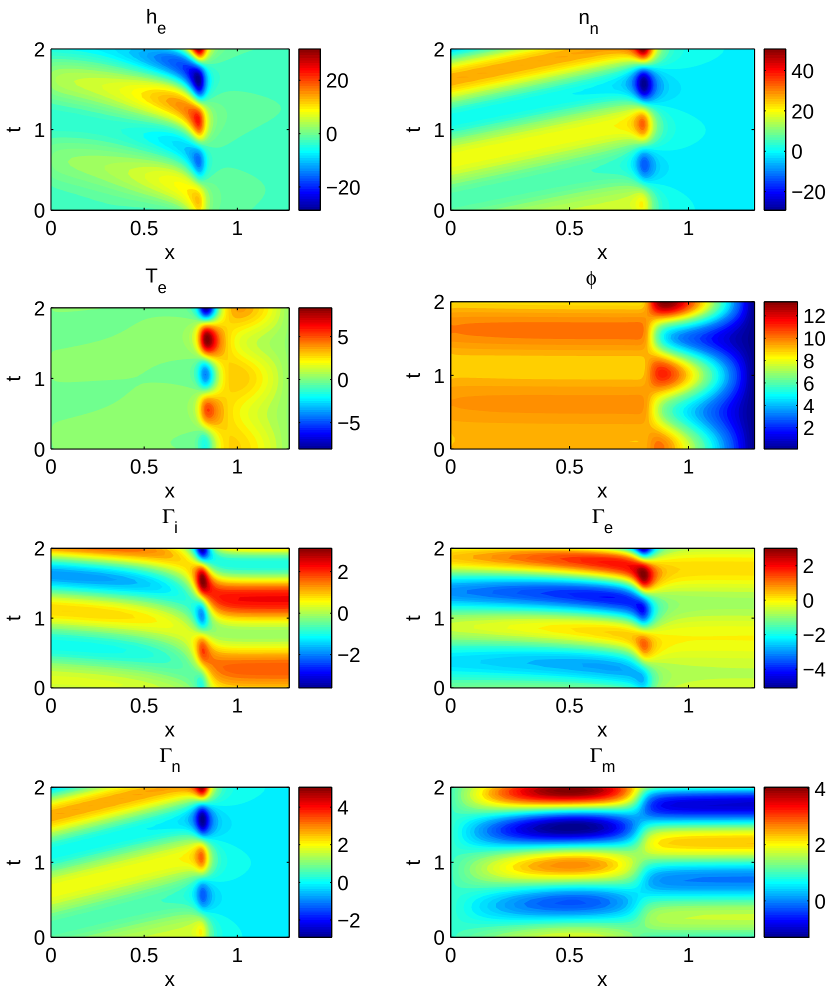

Figure 2. Temporal variation of a self-excited mode in non-dimensional variables. Time t, axial coordinate x, plasma density n, neutral density n_n, electron temperature T_e, electric potential ϕ, ion flux Γ_i, electron flux Γ_e, neutral flux Γ_n, mass flux Γ_m

V. Conclusions

A linearized fluid two-dimensional model for Hall Effect Thrusters has been presented in this paper. The aim of the model is to determine the mechanism behind the spoke oscillation. Previous analyses, both experimental and theoretical, seem to indicate an important role of the ionization. To this end, the model of Ahedo *et al.* has been extended to two dimensions and linearized to study the response of the steady state solution against small perturbations in the azimuthal direction. The main challenges to solve the resulting differential eigenvalue problem have been highlighted. Some preliminary results for axial oscillations (i.e., breathing mode) have been presented as proof of the potentiality of the model. These preliminary results are in agreement with previous results obtained for the breathing mode. The work presented in this paper is still in progress. In the near future, the influence of the azimuthal wave number on the perturbations must be analyzed. This analysis shall determine whether self-excited modes are possible with properties similar to the experimentally measured spoke and, more importantly, whether those azimuthal oscillations enhance the electron conductivity as measured in experiments. The mechanism behind the growth of those self-excited modes is also of interest together with possible ways to remove them.

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